

1. REAL NUMBERS

§1.1. The History of Number

The order in which you learnt about numbers matches the historical development of the subject. It's amazing to think that in a few years you followed a path that historically took millennia for mankind to walk.

In the beginning were the counting numbers: 1, 2, 3, ... There were many different systems of notation but we won't



go into that here. Numbers were used to count discrete objects and so these numbers were quite adequate. Addition and multiplication were defined. These were used in accounting, one of the earliest applications of arithmetic.

Subtraction and division were also defined, but they could only be carried out in certain cases. You can subtract 5 from 10 but not the other way around. You can divide 20 by 2, or by 5, but not by 3 or 7. The number 20 cannot be divided exactly by 3, but we often say that that it three goes into 20 six times with two left over. If a man had 20 sheep and three sons he could share them fairly by giving them each 6 sheep and leaving two over.

The modern solution of giving them each $6\frac{2}{3}$ sheep would not have occurred to them! But if the man had 20

cheeses, large round cylinders of cheese, it would be feasible to cut the remaining two cheeses into thirds and give each son two of these thirds in addition to the six whole ones. And so, fractions were born.

The Greeks had a well-developed sense of Geometry and they considered numbers to represent lengths, instead of just counting things. The fractions (we would call them positive rational numbers) seemed to be adequate for this purpose. This was until Pythagoras' Theorem was discovered. The square of the hypotenuse is the sum of the squares on the other two sides. This meant that a right-angled triangle, with sides measuring 1 cubit each (cubits were an ancient unit of measurement) would have a hypotenuse measuring exactly $\sqrt{2}$ cubits long. This was perplexing because they were able to prove that no rational number gives exactly 2 when you square it. The following is not their original proof, but it will do for our purposes.

Theorem 1: There is no rational number $\frac{m}{n}$ whose square is 2.

Proof: Suppose that $\left(\frac{m}{n}\right)^2 = 2$. Then $m^2 = 2n^2$. Now every number can be factorised uniquely into primes. And a perfect square must have an even number of factors of 2. But $2n^2$ must have an odd number of factors of 2 and so can't be a perfect square. 🙌😊

At first they treated these irrational numbers with mistrust, but in time they were accepted as normal numbers. And so we had the system of positive real numbers. Mankind learnt to add, subtract, multiply and divide fractions by fractions.

I don't know whether you realise it but the arithmetic of fractions is extremely sophisticated. This is because the same fraction can be represented using many different pairs of counting numbers. For example, $\frac{3}{4} = \frac{6}{8} = \frac{15}{20} = \frac{39}{52} = \dots$ The rule for adding fractions involves doing some calculation with these counting numbers. What if we got a different answer if we added $\frac{3}{4} + \frac{10}{14}$ compared to adding $\frac{15}{20} + \frac{5}{7}$?

In the first case our answer is $\frac{42 + 40}{56} = \frac{82}{56}$ while in the second case we get $\frac{105 + 100}{140} = \frac{205}{140}$. Now, although these look different, they both 'cancel down' to $\frac{41}{28}$ and so represent the same fraction. But was this just a lucky coincidence or will it always happen? Your teacher, quite rightly, would have glossed over this potential difficulty. You always got the same answer if you'd used equivalent fractions and you assumed that this was always going to be the case.

Arithmetic continued on for many centuries with fractions and whole numbers being the only numbers that existed. The number zero didn't exist, let alone negative numbers. There seemed to be no need for zero as a counting number. If you wanted to say that "the number of unicorns in Australia is zero" you might more easily say that "there are no unicorns in Australia". And a line of zero length wouldn't be a line at all – just a single point.

It was the advent of place value notation for numbers that revolutionised arithmetic. Imagine doing long division with the numbers written in Roman numerals! But there was a problem with a number such as 503 before zero was invented. You could leave a space and write 5 3, but this would be confusing. The zero symbol was invented in India, and spread to the West via Persia. But 0 wasn't considered a number at first – just a piece of punctuation – rather like the comma in numbers such as 1,000.

Gradually thinking about number changed from regarding it as representing a length to considering it as representing a position on a line relative to some fixed point, called the **origin**. That point was represented by 0, and since you can go left of the origin just as easily as going to the right, negative numbers were born.

Arithmetic had to be extended to cope with these new numbers. But it was discovered that this could be done in such a way that the rules of arithmetic that held for counting numbers still worked for **real numbers** –

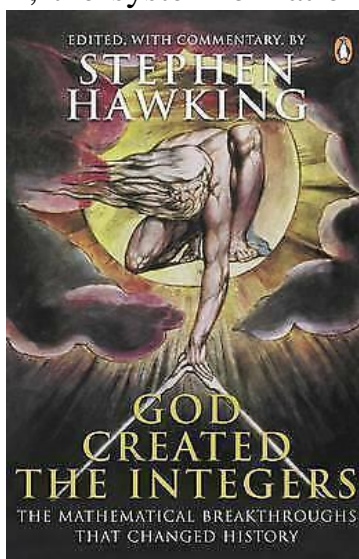
real numbers being those numbers representing points on the number line.

Actually the process didn't stop there. The position of a point in a plane, relative to two perpendicular axes, gave rise to new numbers, called **complex numbers**. But for many decades these extra numbers were considered not to exist, even though using them proved extremely useful. That's why the numbers we learn at school are called *real* numbers.

But as always, what proves to be useful in mathematics is considered to exist and so these non-real complex numbers are now considered to exist just as validly as the numbers 1, 2, 3, ... We'll develop the theory of complex numbers in a later chapter.

For our purposes here we have the system of real number, which we denote by \mathbb{R} , the system of rational numbers (positive, negative or zero fractions) which we denote by \mathbb{Q} (standing for 'quotient') and \mathbb{Z} , the system of integers (positive, negative or zero whole numbers). Why do we use the letter Z? The reason is that the German work for integers is 'Zahlen'.

If we just want to consider the positive real numbers or the positive rational numbers we attach a +



sign after the name, giving \mathbb{R}^+ , \mathbb{Q}^+ and \mathbb{Z}^+ . Sometimes we want to consider the non-zero numbers in each of these systems and so write $\mathbb{R}^\#$, $\mathbb{Q}^\#$ and $\mathbb{Z}^\#$.

The mathematician Kronecker once said “God created the integers; all else is the work of man”. The modern view of mathematics is that it’s irrelevant to ask whether God created various numbers, and whether man discovered them or whether man invented them. In one sense all of mathematics is a human invention. But one gets the sense that what one is inventing works so wonderfully well that we are merely *reinventing* it. Whether there is a God who created the human mind to be capable of such marvellous inventions or whether it has all come about by pure chance, is an interesting theological/philosophical question. However it’s entirely irrelevant to mathematics. So we invent things as needed and don’t trouble ourselves with the question as to whether they *really* exist.

At about the same time that the real numbers came into being their representation by decimals was introduced. Many real numbers require infinitely many decimal places to represent them exactly. (In view of more advanced mathematics we should say that “most numbers have infinite decimal expansions”.) In some of these cases the numbers have *repeating* decimal expansions – that is, after a certain point a block of one or more digits repeats over and over. They can be written

compactly by putting a dot over a repeating digit, or over the first and last digits of a repeating block of digits.

Example 1:

$$\begin{aligned} 2/3 &= 0.6666666666666666\dots\dots\dots = 0.\dot{6} \\ 22/7 &= 3.14285714285714\dots\dots\dots = 3.\dot{1}4285\dot{7} \end{aligned}$$

Numbers ending in $\dot{9}$ can be represented in two ways.

Example 2: $3.56\dot{9} = 3.57$.

Proof: Let $x = 3.569999\dots$

Then $10x = 35.69999\dots$

Now $x = 3.569999$

Subtracting, $9x = 32.13$.

Hence $x = 3.57$.

It's often argued, wrongly, that $0.9999\dots$ is a tiny bit less than 1. But, using the above argument we can see that it's simply another way of writing 1. If you're still not convinced ask yourself what lies half way between $0.9999\dots$ and 1?

Many (in fact, most) real numbers have infinite decimal expansions that do not repeat. For example the number π is the circumference of a circle whose diameter is 1. This number is so important in mathematics we use a special symbol for it. Unlike x , π is not a variable. It is

shorthand for a number that would take forever to write out exactly.

Example 3:

$$\pi = 3.14159266358979323846264338327950288.....$$

It can be shown that the decimal expansion of π is not repeating.

It's often claimed that $\pi = 22/7$. This is not true, but it's a fairly good approximation.

$$\text{In fact } 22/7 = 3.142857142857 \dots$$

$$\text{A better approximation is } 355/113 = 3.1415929 \dots$$

Theorem 2: A decimal represents a rational number if and only if it is a repeating decimal.

Proof: Suppose $x = m/n$. If we carry out the process of division the rest of the calculation is determined simply by the remainder. Since there are only n possible remainders when we divide by n the remainders must repeat and the digits will therefore repeat.

Suppose now that:

$$x = 0 . a_1 a_2 \dots a_h d_1 \dots d_k d_1 \dots d_k d_1 \dots$$

$$\text{Then } 10^h x = a_1 a_2 \dots a_h . d_1 \dots d_k d_1 \dots d_k d_1 \dots \text{ and}$$

$$10^{h+k} x = a_1 a_2 \dots a_h d_1 \dots d_k . d_1 \dots d_k d_1 \dots d_k d_1 \dots$$

$$\text{Subtracting we get } 10^{h+k} x - 10^h x = a_1 a_2 \dots a_h d_1 d_2 \dots d_k.$$

Let m be the integer whose decimal expansion is given on the right hand and let $n = 10^h(10^k - 1)$. Then $x = m/n$. 🙌 😊

Example 4: Find the value of $3.723723723\dots$

Solution: Let $x = 3.723723723\dots$

Then $1000x = 3723.723723723\dots$

(The repeating part has length 3 so we multiply by 10^3 .)

Subtracting we get $999x = 3720$ which means that

$$x = \frac{3720}{999} = \frac{1240}{333}.$$

§1.2. The Laws of Algebra



There are certain properties that the real numbers possess which underlie basic algebra.

The Commutative Laws:

For all real numbers:

$$a + b = b + a \text{ and } ab = ba.$$

You've known for a long time that it doesn't matter in which order you add a list of numbers, and you can multiply numbers in any order. But be warned. This is not the usual state of affairs in mathematics, or in life for that matter. The order in which you carry out certain operations is often highly critical. If a watchmaker has to reassemble a watch it *does* matter in which order he puts in the pieces. Carrying out the operations of putting on your socks and your shoes, you get quite a different result if you reverse the usual order!

There are mathematical objects which you can multiply where $ab \neq ba$. This leads to all sorts of complications. So remember, despite the heading for this section we're not really discussing the laws of algebra – only the laws of the algebra of real numbers.

The Associative Laws:

For all real numbers:

$$(a + b) + c = a + (b + c) \text{ and } (ab)c = a(bc).$$

This means that when we add three or more terms, or multiply three or more factors, it doesn't matter how we group them. As a result we usually write the above as: $a + b + c$ and abc respectively.

These aren't laws that operate in all algebraic situations, though they do hold in most. When you learn about vector products you will discover a system that is not associative.

Because of the associative laws for real numbers we write things like $3x$, meaning $x + x + x$ and $x^3 = xxx$. If addition or multiplication were not associative these would be ambiguous. Does $3x$ mean $(x + x) + x$, or is it $x + (x + x)$? Does x^4 mean $(xx)(xx)$ or $(x(xx))x$ or $((xx)x)x$? Luckily it doesn't matter.

Identities:

There are two real numbers that behave specially when it comes to addition and multiplication. These are the numbers 0 and 1.

For all real numbers:

$$\mathbf{x + 0 = x \text{ and } 1x = x.}$$

The numbers 0 and 1 are called the **additive identity** (0) and the **multiplicative identity** (1).

Inverses:

For all real numbers x there's a number, written $-x$ such that:

$$x + (-x) = 0.$$

For all non-zero real numbers x there is a number, written x^{-1} such that:

$$xx^{-1} = 1.$$

The number $-x$ is called the **additive inverse** of x and x^{-1} is called the **multiplicative inverse** of x .

Why the exception? Why can't 0 have a multiplicative inverse? Why don't we write $\frac{1}{0} = \infty$? Quite apart from the difficulty of finding a point on the real line to represent it, our whole system would implode if we allowed this. This is because we would then have $0\infty = 1$. But what's wrong with that? The answer is "nothing, if you want to abandon the distributive laws". We insist on having the distributive law, so 0 will just have to do without an inverse. Let's see what the distributive law is.

Distributive Laws:

The distributive law is what ties the additive and multiplicative structures of the real numbers together.

For all real numbers:

$$(a + b) c = ac + bc.$$

If $0^\infty = 1$ then $1 = 0^\infty = (0 + 0)^\infty = 0^\infty + 0^\infty = 1 + 1 = 2$, a contradiction.

Because multiplication is commutative we can also write $a(b + c) = ab + ac$. In some mathematical systems where multiplication is not commutative they are different laws and we usually want both to operate. That is why we call them the Distributive Laws. They are fundamental properties of the real numbers and we use them every time we expand an expression.

Example 5: Expand $(3x + 2y)(5x + y)$.

Solution:
$$\begin{aligned}(3x + 2y)(5x + y) &= (3x + 2y)(5x) + (3x + 2y)(y) \\ &= 15x^2 + 10xy + 3xy + 2y^2 \\ &= 15x^2 + 13xy + 2y^2.\end{aligned}$$

Here we're using more than the distributive law. When we multiplied $3x$ by $5x$ to get $15x^2$ we unconsciously made use of the associative law and the commutative law for multiplication. And the fact that we wrote the answer as $15x^2 + 13xy + 2y^2$ and not $(15x^2 + 13xy) + 2y^2$ or $15x^2 + (13xy + 2y^2)$ shows that we're mindful of the associative law for addition. Note too that in writing $10xy + 3xy$ as $13xy$ we are unconsciously making use of the distributive law. If we had to justify it we could write $10xy + 3xy = (10 + 3)xy = 13xy$.

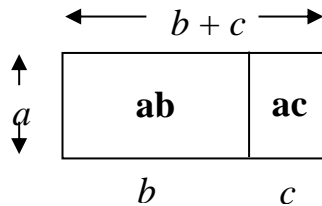
When we worked with algebraic expressions in high school we probably didn't think too much about what we were doing. We just instinctively changed one expression

into an equivalent one. But if we want to know *why* things work the way they do we could justify everything from the above fundamental laws.

And what if we wanted a proof that these laws are correct? To do that we would have to carefully define a real number. Even defining the number 2 in a formal way is quite a sophisticated matter. Now is not the time or place to get bogged down with such fundamentals!

However we can provide a geometric explanation of the distributive law. If you accept that ab is the area of an $a \times b$ rectangle then this picture demonstrates that

$$a(b + c) = ab + ac.$$



Someone interested in the foundations of mathematics would not be content with such a proof, but it will do for now.

Now there are many other laws of algebra that were not given in the above list. For example we all know that “if you multiply by zero you get zero”. Is this yet one more

law we have to accept? No, it is a simple consequence of the ones we've already given.

Theorem 3: For all real numbers x , $0x = 0$.

Solution:

$$\begin{aligned}0x &= (0 + 0)x \\ &= 0x + 0x\end{aligned}$$

We would now cancel $0x$ on both sides to get $0x = 0$.

But what is exactly going on when we cancel?

From $0x = 0x + 0x$ we get, by adding $-0x$ to both sides

$$0x + (-0x) = (0x + 0x) + (-0x)$$

$$\therefore 0 = 0x + (0x + (-0x))$$

$$= 0x + 0$$

$$= 0x. \text{ 🙌 😊}$$

Can you identify the many basic laws we're using here?

Students are often perplexed as to why -1 times -1 is $+1$. They're often fobbed off by teachers with some vague explanation such as "two negatives make a positive". Certainly if something is not impossible then it is possible, but what has this to do with algebra? The more satisfactory explanation is the following.

Theorem 4: $(-1)(-1) = 1$.

Proof: $(-1)(-1) = (-1) + 0$

$$= (-1)(-1) + ((-1) + 1)$$

$$= ((-1)(-1) + (-1)) + 1$$

$$= ((-1)(-1) + (-1)1) + 1$$

$$= (-1)(-1 + 1) + 1$$

$$\begin{aligned}
&= (-1) \cdot 0 + 1 \\
&= 0 + 1 \\
&= 1. \text{ 🙌 😊}
\end{aligned}$$

Theorem 5: If $xy = 0$ then either $x = 0$ or $y = 0$.

Proof: Suppose $xy = 0$. If $x \neq 0$ then x^{-1} exists and we can multiply both sides by x^{-1} to get $x^{-1}(xy) = 0$. By the associative law this becomes $(x^{-1}x)y = 0$.

Hence $1y = y = 0$. 🙌 😊

Theorem 6 (CANCELLATION LAW):

If $xy = xz$ and $x \neq 0$ then $y = z$.

Proof: This is proved similarly. 🙌

§1.3. Basic Algebraic Identities

There are certain equations that hold for all variables in the algebra of real numbers. Here are some that are useful to know.

Theorem 7: For all real numbers x, y :

$$(1) (x + y)^2 = x^2 + 2xy + y^2;$$

$$(2) (x - y)^2 = x^2 - 2xy + y^2;$$

$$(3) (x - y)(x + y) = x^2 - y^2;$$

$$(4) (x - y)(x^2 + xy + y^2);$$

$$(5) (x + y)(x^2 - xy + y^2).$$

Proof:

$$\begin{aligned}
(1) (x + y)^2 &= (x + y)(x + y) \\
&= x(x + y) + y(x + y) \\
&= x^2 + xy + yx + y^2
\end{aligned}$$

$$\begin{aligned}
 &= x^2 + xy + xy + y^2 \\
 &= x^2 + 2xy + y^2.
 \end{aligned}$$

(2) to (5) can be proved similarly. 🙌😊

The special cases of (3), (4) and (5), where $y = 1$, should be noted as they arise frequently.

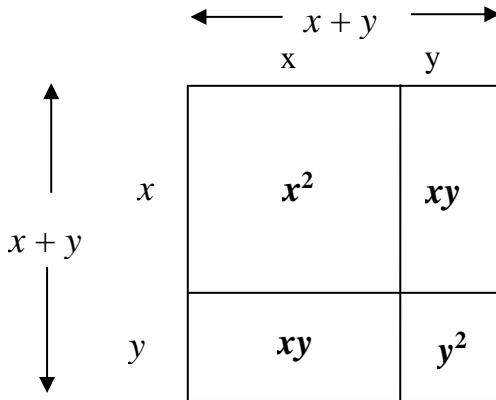
$$x^2 - 1 = (x - 1)(x + 1);$$

$$x^3 - 1 = (x - 1)(x^2 + x + 1);$$

$$x^3 + 1 = (x + 1)(x^2 - x + 1).$$

Also note that although $x^2 - 1$ factorises, $x^2 + 1$ does not.

The first of these identities can be illustrated geometrically as follows.



Example 10: Factorise $x^4 - 1$.

Solution: $x^4 - 1 = (x^2 - 1)(x^2 + 1)$ by replacing x by x^2 in the factorisation of $x^2 - 1$

$$= (x - 1)(x + 1)(x^2 + 1).$$

§1.4. Solving Linear Equations

A **linear equation** (in one variable) has the form:

$$ax + b = c.$$

It's very easy to solve.

Example 11: Solve $7x + 5 = 9$.

Solution: Subtract 5 from both sides, giving $7x = 4$.
Divide both sides by 7, giving $x = 4/7$.

Even though we'd always give the above one line solution there's a lot that lies beneath the surface. Just this once let's proceed more slowly, justifying each step carefully.

Suppose $7x + 5 = 9$.

Then $(7x + 5) + (-5) = 9 + (-5) = 4$.

Using the associative law for addition this becomes:

$$7x + (5 + (-5)) = 4.$$

Using the law of inverses under addition this becomes:

$$7x + 0 = 4.$$

Using the law of additive identity this becomes $7x = 4$.

The simple act of subtracting 5 from both sides uses several of the fundamental laws of algebra.

Now we multiply both sides by 7^{-1} to get:

$$7^{-1}(7x) = 7^{-1} \cdot 4 = \frac{4}{7}.$$

We now have to appeal to the associative law for multiplication to deduce that $(7^{-1}7)x = \frac{4}{7}$, and using the

laws for multiplicative inverses and identity this gives us $x = \frac{4}{7}$. But strictly speaking we haven't finished.

We've only found the non-solutions! We supposed that x is a solution and deduced that if there is a solution it would have to be $\frac{4}{7}$. But, what if the equation has *no* solution. Logically, all we've done is to rule out anything other than $\frac{4}{7}$.

To complete the exercise we need to verify that this is indeed a solution, by starting with $x = \frac{4}{7}$ and working back to $7x + 5 = 8$.

Suppose $x = \frac{4}{7} = 7^{-1}.4$.

Then $7x = 7(7^{-1}.4)$

$$= (7.7^{-1})4 = 1.4 = 4, \text{ using various laws.}$$

So $7x + 5 = 4 + 5 = 9$. Yes, we indeed have a solution, namely $x = 4/7$.

If this sounds like we are being overly pedantic, then that's true – for simple equations like this. What works here in one direction also works in the other. But beware! There are equations which, when we solve in the normal way, we get a number of possible solutions. But when we

test them not all of them may work. The ones that don't we call **spurious solutions**.

There's nothing wrong with the logic, however. What we usually call "solving an equation" is simply a case of narrowing down the possibilities to a small number, which, in theory, we then should then check. But we get used to particular types of equations where we know that there are no spurious solutions and we can often write down the solution, or solutions at once.

§1.5. Quadratic Equations

A **quadratic equation** has the form $ax^2 + bx + c = 0$ where a, b, c are constants (real numbers) and where $a \neq 0$. A common way to solve a quadratic is to factorise the left-hand side.

Example 12: Solve $x^2 + 5x + 6 = 0$.

Solution: We can write $x^2 + 5x + 6$ as $(x + 2)(x + 3)$ and so we now have to solve the equation $(x + 2)(x + 3) = 0$.

Now if a product of two real numbers is zero, at least one of them must be zero.

So $x + 2 = 0$ or $x + 3 = 0$, which gives $x = -2$ and -3 as the two solutions.

(We should either check that both of these work, or glance over our solution to assure ourselves that every step we performed is reversible.)

The hardest part of this method is in the factorising. Here, because the coefficient of x^2 is 1 we simply need to find two numbers whose sum is 5 and whose product is 6. Where the coefficient is something other than 1 it's a little more difficult.

Example 13: Solve $30x^2 - 103x + 70 = 0$.

Solution: $30x^2 - 103x + 70 = (2x - 5)(15x - 14) = 0$ so $x = 5/2$ or $14/15$.

How did we go about factorising the quadratic? We looked for factors of 30 and of 70 that combine in the right way to give 103. Perhaps $(5x + 7)(6x + 10)$? No, that gives $92x$.

What about $(5x + 14)(6x + 5)$? No, that gives $109x$.

It seems like we are forced to do 'trial an error'. It could be that the quadratic doesn't factorise nicely, with whole numbers.

Factorising a quadratic is a good method for solving a quadratic equation provided we can spot the factorisation quickly. But there's a general method that will always work – the **quadratic equation formula**.

Theorem 8: If $b^2 - 4ac \geq 0$, the solutions to the quadratic $ax^2 + bx + c = 0$ are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Proof: Suppose $ax^2 + bx + c = 0$.

$$\text{Then } x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

$$\text{Now } \left(x + \frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}.$$

$$\text{Hence } \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0.$$

$$\text{So } \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Taking square roots this gives us:

$$\begin{aligned} x + \frac{b}{2a} &= \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \\ &= \pm \frac{\sqrt{b^2 - 4ac}}{2a}. \end{aligned}$$

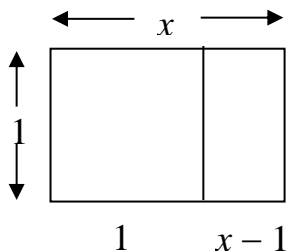
The quadratic formula now follows easily.

This proof is called **Completing the Square**. While it is a simple technique it doesn't generalise to cubics and quartics. A more instructive proof will be given in chapter 10, which highlights the role that symmetry plays in the solution of polynomial equations.

If $b^2 - 4ac < 0$ we'd need the square root of a negative number which, as far as we are concerned at this stage, doesn't exist and so there are no solutions. More correctly, there are no *real* solutions in such a case. We will later learn about complex numbers, a system of numbers that includes all the real numbers but also includes square roots for negative numbers.

Example 14: Find the proportions of a rectangle such that if you cut off a square at one end, the left-over rectangle would have the same proportions as the original one.

Solution: Let the smaller side have length 1 and the longer side have length x .



The larger rectangle has length x and breadth 1, while the smaller rectangle has length 1 and breadth $x - 1$.

$$\text{So } \frac{x}{1} = \frac{1}{x-1}.$$

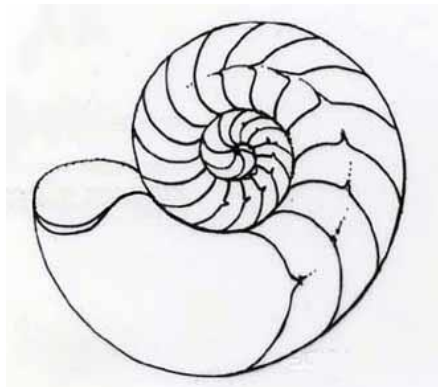
Hence $x(x - 1) = 1$ and so $x^2 - x - 1 = 0$.

This doesn't factorise, but using the quadratic formula we get:

$$x = \frac{1 \pm \sqrt{5}}{2}.$$



Now $\frac{1-\sqrt{5}}{2} < 0$ and although it is a solution to the



quadratic equation it clearly can't be a solution to the original problem. So the ratio of the longer side to the shorter one for such a rectangle is $\frac{1+\sqrt{5}}{2}$.

This number, about 1.868, is called the

Golden Mean. It is supposed to be the ideal proportion, aesthetically, for a rectangle, and many architects, including the builders of the Parthenon, have used this proportion in their designs.

The golden mean also occurs in Nature, showing that the Divine Architect must be able to solve quadratic equations!

§1.6. Sum and Product of Zeros

The **zeros** of a quadratic expression are the solutions of the corresponding equation. So, the zeros of $x^2 - 5x + 6$ are 2, 3 because the solutions of $x^2 - 5x + 6 = 0$ are 2, 3.

If α and β are the zeros of the quadratic $ax^2 + bx + c$ then we can express $\alpha + \beta$ and $\alpha\beta$ in terms of the coefficients a, b, c .

Theorem 8: If α and β are the zeros of the quadratic $ax^2 + bx + c$ then:

$$\alpha + \beta = -\frac{b}{a} \text{ and}$$

$$\alpha\beta = \frac{c}{a}.$$

Proof: While we could prove these by using the quadratic formula, the simplest proof comes from equating

$$ax^2 + bx + c \text{ to}$$

$$a(x - \alpha)(x - \beta) = ax^2 - a(\alpha + \beta)x + a\alpha\beta.$$

Since corresponding coefficients must be equal we have $b = -a(\alpha + \beta)$ and $c = a\alpha\beta$. 🙌😊

Functions in α and β that are symmetric can be expressed in terms of $\alpha + \beta$ and $\alpha\beta$ and hence can be expressed easily in terms of the coefficients.

Example 15: If α, β are the roots of the quadratic $x^2 - x - 2$ find the values of:

$$(i) \alpha^2 + \beta^2; \quad (ii) \frac{1}{\alpha} + \frac{1}{\beta};$$

$$(iii) \alpha^2\beta + \alpha\beta^2.$$

Solution: $\alpha + \beta = 1$ and $\alpha\beta = -2$.

$$(i) \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 1^2 + 4 = 5.$$

$$(ii) \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = -\frac{1}{2}.$$

$$(iii) \alpha^2\beta + \alpha\beta^2 = \alpha\beta(\alpha + \beta) = -2.$$

§1.7. Fractions

A fraction is an expression of the form $\frac{m}{n}$. We sometimes write it as m/n .

The number on the top is called the **numerator** and the number on the bottom is called the **denominator**. When the numerator and denominator are integers, we call the fraction a **rational number**.

Adding and subtracting fractions, whether they are algebraic, or simply arithmetic, can be quite difficult, though, if the denominators are the same it is very easy.

Example 16: $\frac{3}{17} + \frac{4}{17} = \frac{7}{17}$.

Adding seventeenths is no more difficult than adding apples. Three plus four more is seven of them.

Example 17: $\frac{x}{x^2 + 1} + \frac{1}{x^2 + 1} = \frac{x + 1}{x^2 + 1}$.
 $\frac{x^2}{x^2 + 1} + \frac{1}{x^2 + 1} = \frac{x^2 + 1}{x^2 + 1} = 1$.

In this last case we were able to simplify the fraction by dividing the numerator and denominator by the same expression.

When the denominators are different we have to find a common multiple, preferably the *least* common multiple, though that is not absolutely necessary.

Example 18: Simplify $\frac{2}{15} - \frac{3}{20}$.

Solution: Here the least common multiple of the denominators is 60.

$$\frac{2}{15} - \frac{3}{20} = \frac{8}{60} - \frac{9}{60} = -\frac{1}{60}.$$

However, if we simply used the product of the denominators, 300, we'd eventually get the same answer.

$$\frac{2}{15} - \frac{3}{20} = \frac{40}{300} - \frac{45}{300} = -\frac{5}{300} = -\frac{1}{60}.$$

Example 19: Simplify $\frac{x}{x^2 - 1} - \frac{1}{x^2 + 2x + 1}$.

Solution: $x^2 - 1 = (x - 1)(x + 1)$ and $x^2 + 2x + 1 = (x + 1)^2$. Hence both divide $(x^2 - 1)(x + 1)$.

$$\begin{aligned} \text{So } \frac{x}{x^2 - 1} - \frac{1}{x^2 + 2x + 1} &= \frac{x(x + 1)}{(x^2 - 1)(x + 1)} - \frac{x - 1}{(x^2 - 1)(x + 1)} \\ &= \frac{x^2 + 1}{(x^2 - 1)(x + 1)}. \end{aligned}$$

Multiplying fractions is much easier than adding or subtracting them. You simply multiply the numerators and multiply the denominators. And to divide fractions

you “invert and multiply”. The fraction that is being divided by has its numerator and denominator swapped to form its multiplicative inverse.

Example 20: Simplify $\frac{18}{35} \times \frac{20}{27}$.

$$\frac{18}{35} \times \frac{20}{27} = \frac{360}{945} = \frac{8}{21} \text{ after ‘cancelling down’}.$$

But what is much simpler is to do the cancelling before the final multiplications.

$$\begin{aligned} \frac{18}{35} \times \frac{20}{27} &= \frac{2}{35} \times \frac{20}{3}, \text{ cancelling by 9} \\ &= \frac{2}{7} \times \frac{4}{3}, \text{ cancelling by 5} \\ &= \frac{8}{21}. \end{aligned}$$

It helps, in algebraic cases, to factorise the numerators and denominators first.

Example 21: Simplify $\frac{x^2 + 1}{x^3 + 1} \times \frac{x + 1}{x^3 + x}$.

Solution:

$$\begin{aligned} x^3 + 1 &= (x + 1)(x^2 - x + 1) \text{ and } x^3 + x = x(x^2 + 1). \\ \frac{x^2 + 1}{x^3 + 1} \times \frac{x + 1}{x^3 + x} &= \frac{x^2 + 1}{(x + 1)(x^2 - x + 1)} \times \frac{x + 1}{x(x^2 + 1)}. \\ &= \frac{1}{(x + 1)(x^2 - x + 1)} \times \frac{x + 1}{x} \\ &= \frac{1}{x(x^2 - x + 1)}. \end{aligned}$$

Example 22: Simplify $\frac{18}{55} \div \frac{16}{25}$.

Solution:
$$\begin{aligned}\frac{18}{55} \div \frac{16}{25} &= \frac{18}{55} \times \frac{25}{16} \\ &= \frac{9}{11} \times \frac{5}{8}, \text{ cancelling by 2 and 5} \\ &= \frac{45}{88}.\end{aligned}$$

Example 23: Simplify $\frac{x^4 - 1}{x^4 + 1} \div \frac{x^2 + 1}{x + 1}$.

Solution:
$$\begin{aligned}\frac{x^4 - 1}{x^4 + 1} \div \frac{x^2 + 1}{x + 1} \\ &= \frac{x^4 - 1}{x^4 + 1} \times \frac{x + 1}{x^2 + 1} \\ &= \frac{(x - 1)(x + 1)(x^2 + 1)}{x^4 + 1} \times \frac{x + 1}{x^2 + 1} \\ &= \frac{(x - 1)(x + 1)^2}{x^4 + 1}, \text{ cancelling by } x^2 + 1.\end{aligned}$$

§1.8. Surd Equations

A **surd** is a square root, cube root etc. An equation involving surds can often be solved by squaring, or raising to some other power, both sides of the equation to get rid of the surds. But beware! In the process of squaring we may pick up spurious solutions that don't satisfy the original equation.

Example 24: Solve the equation $\sqrt{5x - 6} = x$.

Solution: Square both sides to get $5x - 6 = x^2$.

$$\therefore x^2 - 5x + 6 = 0.$$

$$\therefore (x - 2)(x - 3) = 0.$$

$$\therefore x = 2 \text{ or } x = 3.$$

But, in the process of squaring, we may pick up ‘solutions’ that don’t satisfy the original equation. We must check our answers in the original surd equation. In example 24 they both work but this is not always the case.

Example 25: Solve the equation $\sqrt{3x - 12} - \sqrt{x - 3} = 1$.

Solution: Suppose $\sqrt{3x - 12} - \sqrt{x - 3} = 1$. Add $\sqrt{x - 3}$ to both sides.

Then $\sqrt{3x - 12} = \sqrt{x - 3} + 1$. Now square both sides.

$$\therefore 3x - 12 = (x - 3) + 1 + 2\sqrt{x - 3} \quad \dots\dots\dots (1)$$

$$\therefore 2\sqrt{x - 3} = 2x - 10.$$

$\therefore \sqrt{x - 3} = x - 5$. Now square both sides.

$$\therefore x - 3 = (x - 5)^2 \quad \dots\dots\dots (2)$$

$$= x^2 - 10x + 25.$$

$$\therefore x^2 - 11x + 28 = 0.$$

$$\therefore (x - 4)(x - 7) = 0.$$

$\therefore x = 4$ or 7 . These are the potential solutions.

If $x = 4$ then $\sqrt{3x - 12} - \sqrt{x - 3} = -1$ and so is not a solution to the original equation. It crept in when we squared both sides at step (1).

If $x = 7$ then $\sqrt{3x - 12} - \sqrt{x - 3} = 3 - 2 = 1$, so this is a valid solution.

So $x = 7$ is the *only* solution.

Example 26: Solve the equation $\sqrt{8 - x} + \sqrt{x - 3} = 1$.

Solution: Suppose $\sqrt{8 - x} + \sqrt{x - 3} = 1$.

Then $\sqrt{8 - x} = 1 - \sqrt{x - 3}$. Now square both sides.

$$\therefore 8 - x = (x - 3) + 1 - 2\sqrt{x - 3} \quad \dots\dots\dots (1)$$

$$\therefore 2\sqrt{x - 3} = 2x - 10.$$

$\therefore \sqrt{x - 3} = x - 5$. Now square both sides.

$$\begin{aligned} \therefore x - 3 &= (x - 5)^2 && \dots\dots\dots (2) \\ &= x^2 - 10x + 25. \end{aligned}$$

$$\therefore x^2 - 11x + 28 = 0.$$

$$\therefore (x - 4)(x - 7) = 0.$$

$\therefore x = 4$ or 7 . These are the potential solutions.

If $x = 4$ then $\sqrt{8 - x} + \sqrt{x - 3} = 2 + 1 = 3$ and so $x = 4$ is not a solution to the original equation. It crept in when we squared both sides at step (1).

If $x = 7$ then $\sqrt{8 - x} + \sqrt{x - 3} = 1 + 2 = 3$ and so $x = 7$ is also not a solution.

So this equation has no solutions.

EXERCISES FOR CHAPTER 1

Exercise 1: Simplify $\frac{x-1}{x^2-4} \div \left(\frac{x}{x+1} + \frac{1}{x-3}\right)$.

Exercise 2:

Solve the quadratic equation $x^2 - 3x - 40 = 0$.

Exercise 3:

Solve the quadratic equation $x^2 - 2x - 40 = 0$.

Exercise 4:

Solve the quadratic equation $x^2 - 2x + 40 = 0$.

Exercise 5: If α, β are the roots of the quadratic $2x^2 + 7x - 3$, find:

(i) $\alpha^2 + \beta^2$;

(ii) $\alpha^5\beta^2 + \alpha^2\beta^5$;

Exercise 6: Solve the equation $\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1} = 0$.

Exercise 7: Solve the equation $\sqrt{x+1} + \sqrt{9-x} = 4$.

Exercise 8: Solve the equation $\sqrt{x^2-1} + \sqrt{x+2} = x$.

Exercise 9: Solve the equation $\sqrt{x+2} + \sqrt{2x+3} = 4$.

Exercise 10: Solve the equation $\sqrt{x+4} + \sqrt{2x-9} = 4$.

SOLUTIONS FOR CHAPTER 1

Exercise 1: $\frac{x^2 - x + 1}{x^2 - 4} \div \left(\frac{x}{x+1} + \frac{1}{x-2} \right)$

$$= \frac{x^2 - x + 1}{x^2 - 4} \div \left(\frac{x^2 - 2x + x + 1}{(x+1)(x-2)} \right)$$
$$= \frac{x^2 - x + 1}{(x-2)(x+2)} \times \frac{(x+1)(x-2)}{x^2 - x + 1} = \frac{x+1}{x+2}.$$

Exercise 2: $(x+5)(x-8) = 0$, so $x = -5$ or 8 .

Exercise 3: This doesn't factorise nicely so we use the quadratic equation formula.

$$x = \frac{2 \pm \sqrt{4 + 160}}{2} \approx -5.403 \text{ and } 7.403.$$

Exercise 4: We use the quadratic formula to get

$x = \frac{2 \pm \sqrt{4 - 160}}{2}$. At this stage we're only looking for real solutions. We have yet to learn about complex numbers. So we'd say that $\sqrt{-156}$ "does not exist" and therefore the quadratic "has no solutions".

Exercise 5: $\alpha + \beta = -\frac{7}{2}$ and $\alpha\beta = -\frac{3}{2}$.

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{49}{4} + 3 = \frac{61}{4}$.

(ii) $\alpha^5\beta^2 + \alpha^2\beta^5 = \alpha^2\beta^2(\alpha^3 + \beta^3)$.

Now $\alpha^2\beta^2 = \frac{9}{4}$ and $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha^2\beta - 3\alpha\beta^2$
 $= (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
 $= -\frac{343}{8} - \frac{63}{4} = -\frac{469}{8}$.

Hence $\alpha^5\beta^2 + \alpha^2\beta^5 = -\frac{9}{4} \cdot \frac{469}{8} = -\frac{4221}{32}$.

Exercise 6: $\frac{(x^2 - 1) + (x^2 + x) + (x^2 - x)}{x(x^2 - 1)} = 0$.

Hence $\frac{3x^2 - 1}{x(x^2 - 1)} = 0$ and so $x^2 = \frac{1}{3}$ or $x = \pm \frac{1}{\sqrt{3}}$.

Exercise 7: Suppose $\sqrt{x+1} + \sqrt{9-x} = 4$.

Then $\sqrt{9-x} = 4 - \sqrt{x+1}$.

Squaring we get $9 - x = 16 + (x+1) - 8\sqrt{x+1}$.

Hence $8\sqrt{x+1} = 2x + 8$ and so $4\sqrt{x+1} = x + 4$.

Square again. $16x + 16 = x^2 + 8x + 16$.

Hence $x^2 - 8x = 0$, which gives $x = 0$ and $x = 8$.

Checking in the original equation we find that both are solutions.

Exercise 8: Suppose $\sqrt{x^2 - 1} + \sqrt{x + 2} = x$.

Then $\sqrt{x^2 - 1} = x - \sqrt{x + 2}$.

Squaring we get $x^2 - 1 = x^2 - 2\sqrt{x+2} + (x+2)$.

Hence $2\sqrt{x+2} = x+3$.

$$\therefore 4x + 8 = x^2 + 6x + 9.$$

$$\therefore x^2 + 2x + 1 = 0.$$

$$\therefore (x+1)^2 = 0.$$

$$\therefore x = -1.$$

Checking in the original equation we find that $x = -1$ is not a solution, and so the equation has no solutions.

Exercise 9: Suppose $\sqrt{x+2} + \sqrt{2x+3} = 4$.

Then $\sqrt{2x+3} = 4 - \sqrt{x+2}$.

Squaring we get $2x+3 = 16 + (x+2) - 8\sqrt{x+2}$.

Hence $8\sqrt{x+2} = 15 - x$.

$$\therefore 64x + 128 = 225 - 30x + x^2.$$

$$\therefore x^2 - 94x + 97 = 0.$$

$$\therefore x = \frac{94 \pm \sqrt{8448}}{2} = 92.956 \text{ and } 1.0435 \text{ (approximately)}$$

Checking in the original equation we find that $x = 1.0435$ is the only solution.

Exercise 10: Suppose $\sqrt{x+4} + \sqrt{2x-9} = 4$.

Subtract $\sqrt{x+4}$ from both sides and square.

$$\therefore \sqrt{2x-9} = 4 - \sqrt{x+4}.$$

$$\therefore 2x - 9 = 16 + (x+4) - 8\sqrt{x+4}.$$

$$\therefore x - 29 = -8\sqrt{x+4}. \text{ Now square both sides again.}$$

$$\therefore (x-29)^2 = 64(x+4).$$

$$\therefore x^2 - 58x + 841 = 64x + 256.$$

$$\therefore x^2 - 122x + 585.$$

$$\therefore (x - 5)(x - 117) = 0.$$

$$\therefore x = 5 \text{ or } 117.$$

So we have two solutions, right? No! We must check them in the original equation.

If $x = 5$ then $\sqrt{x+4} + \sqrt{2x-9} = 3 + 1 = 4$. This works.

If $x = 117$ then $\sqrt{x+4} + \sqrt{2x-9} = 26$. This is what is sometimes called a “spurious solution”.

What happened when we “solved” the equation was that we showed that *if* there is a solution it has to be either 5 or 117. It doesn’t guarantee that either or both are indeed solutions. Only checking the values at the end does this.

Let’s check at what stage $x = 117$ sneaks in. It doesn’t satisfy the equation

$$x - 29 = -8\sqrt{x+4}$$

although it does satisfy the similar equation

$$x - 29 = 8\sqrt{x+4}$$

Notice that both equations get bundled up together when we square.